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The Power Spectrum for a Multi-Component Inflaton to Second-Order Corrections in the Slow-Roll Expansion

Jin-Ook Gong* Ewan D. Stewart†

Department of Physics, KAIST, Daejeon 305-701, South Korea

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Abstract

We derive the power spectrum $\mathcal{P}(k)$ of the density perturbations produced during inflation up to second-order corrections in the standard slow-roll approximation for an inflaton with more than one degree of freedom. We also present the spectral index n up to first-order corrections including previously missing terms, and the running $dn/d\ln k$ to leading order.

*devourer@muon.kaist.ac.kr

†ewan@kaist.ac.kr

1 Introduction

Inflation [1] generates the primordial perturbations which are the origin of the rich structures, such as galaxies and clusters of galaxy clusters, observed in the universe today. The power spectrum of these perturbations is constrained to be approximately scale invariant by a combination of the measurements of anisotropies on large and small angular scales, as well as galaxy surveys [2]. Moreover, forthcoming measurements such as the MAP and Planck satellites and the Sloan Digital Sky Survey will probe the power spectrum with greater accuracy, lowering the observational errors further. Planck, for example, is expected to be able to measure the power spectrum of the cosmic microwave background with an error of a few percent [3]. Thus, it is very important to calculate the power spectrum precisely, so that we can make use of the observations fully.

The calculation of the density perturbations for an inflaton with a single degree of freedom has been studied for a long time [4, 5, 6], and there have been extensive works on the multi-component inflaton case, especially recently [7, 8, 9, 10]. Among these results, those with first-order corrections [5, 9] are expected to have errors of the order of one percent or smaller for the power spectrum [11], which is comparable with the expected errors in the planned observations. Thus we should reduce the errors and improve the accuracy of the calculations. However, the only existing result including second-order corrections [6] considered only the single-component case. Therefore, it is needed to consider the multi-component inflaton with higher order corrections.

In this paper, we extend our previous formalism [6] to the multi-component case, and calculate the power spectrum up to second-order corrections in the standard slow-roll expansion[1]. We also present the spectral index n up to first-order corrections with new terms inconsistently neglected in the previous result [9], and the running $dn/d\ln k$ to leading order for the first time. It is straightforward to check that our results reduce to those of Ref. [6] in the single component case.

2 The Spectrum for a Multi-Component Inflaton

The power spectrum $\mathcal{P}(k)$ is defined by

$$\frac{2\pi^2}{k^3}\,\mathcal{P}(k)\,\delta^{(3)}(\mathbf{k}-\mathbf{l}) = \langle \mathcal{R}_\mathrm{c}(\mathbf{k})\,{\mathcal{R}_\mathrm{c}}^\dagger(\mathbf{l})\rangle\,, \tag{1}$$

where $\langle\,\rangle$ denotes the vacuum expectation value and $\mathcal{R}_\mathrm{c}$ is the intrinsic curvature perturbation of the comoving hypersurfaces. In the single-component case, the standard result for the power spectrum of the curvature perturbation in the slow-roll approximation is

$$\mathcal{P}(k) \simeq \left(\frac{H}{2\pi}\right)^2 \left(\frac{H}{\dot\phi}\right)^2 . \tag{2}$$

The multi-component case is more subtle. Detailed arguments are given in [8]; here we just give the principal idea. Let $N = \int H dt$ be the number of e-folds of expansion. Then,

[1]For a more general slow-roll expansion, see Ref. [12].

on super-horizon scales, we obtain [8]

$$\Delta\mathcal{R} = \delta N\,. \tag{3}$$

That is, the change in the curvature $\mathcal{R}$ between the initial and final hypersurfaces is given by the perturbation in the number of e-folds of expansion. In particular, if we choose the initial hypersurface to be flat, i.e. $\mathcal{R}(t_1) = 0$, and the final one to be comoving, we get

$$\mathcal{R}_{\rm c}(t_2) = \delta N\,. \tag{4}$$

We take t_1 to be some time during slow-roll inflation, and t_2 to be some time after inflation when $\mathcal{R}_{\rm c}$ has become constant. The relevant scale is assumed to be still outside the horizon at t_2. Now, N depends on both $\phi(t_1)$ and $\dot\phi^{\mathbf{a}}(t_1)$, but as t_1 is during slow-roll inflation, it will be convenient to express this dependence in terms of $\phi(t_1)$ and $\dot\phi^{\mathbf{a}}_\perp(t_1)$, where $\dot\phi^{\mathbf{a}}_\perp \equiv \dot\phi^{\mathbf{a}} - \dot\phi^{\mathbf{a}}_{\rm sr}$ is the deviation of $\dot\phi^{\mathbf{a}}$ from its value $\dot\phi^{\mathbf{a}}_{\rm sr}$ on the exact slow-roll trajectory passing through ϕ. Then we can write[2]

$$\delta N = \frac{\partial N}{\partial\phi^{\mathbf{a}}}\,\delta\phi^{\mathbf{a}}(t_1) + \frac{\partial N}{\partial\dot\phi^{\mathbf{a}}_\perp}\,\delta\dot\phi^{\mathbf{a}}_\perp(t_1)\,. \tag{5}$$

However, $\delta\dot\phi^{\mathbf{a}}_\perp$ is negligible on super-horizon scales during slow-roll inflation. Therefore we get

$$\mathcal{R}_{\rm c}(t_2) = \delta N = \frac{\partial N}{\partial\phi^{\mathbf{a}}}\,\delta\phi^{\mathbf{a}}(t_1)\,. \tag{6}$$

Using Fourier expansion and substituting into Eq. (1), we obtain the power spectrum for a multi-component inflaton

$$\frac{2\pi^2}{k^3}\,\mathcal{P}(k)\,\delta^{(3)}(\mathbf{k}-\mathbf{l}) = \frac{\partial N}{\partial\phi^{\mathbf{a}}}\frac{\partial N}{\partial\phi^{\mathbf{b}}}\left\langle\delta\phi^{\mathbf{a}}(\mathbf{k})\,\delta\phi^{\mathbf{b}\dagger}(\mathbf{l})\right\rangle\,. \tag{7}$$

Now, to leading order in the slow-roll approximation, [8]

$$\left\langle\delta\phi^{\mathbf{a}}(\mathbf{k})\,\delta\phi^{\mathbf{b}\dagger}(\mathbf{l})\right\rangle \simeq \frac{H^2}{2k^3}\,\delta^{(3)}(\mathbf{k}-\mathbf{l})\,h^{\mathbf{ab}}\Big|_{aH=k}\,. \tag{8}$$

Therefore

$$\mathcal{P}(k) \simeq \left(\frac{H}{2\pi}\right)^2 h^{\mathbf{ab}}\frac{\partial N}{\partial\phi^{\mathbf{a}}}\frac{\partial N}{\partial\phi^{\mathbf{b}}}\bigg|_{aH=k}\,. \tag{9}$$

Defining $e^N_{\mathbf{a}}$ to be the unit covector in the direction $\partial N/\partial\phi^{\mathbf{a}}$ and using

$$\dot\phi^{\mathbf{a}}\frac{\partial N}{\partial\phi^{\mathbf{a}}} = -H \tag{10}$$

we can write this as

$$\mathcal{P}(k) \simeq \left(\frac{H}{2\pi}\right)^2\left(\frac{H}{e^N_{\mathbf{a}}\dot\phi^{\mathbf{a}}}\right)^2\bigg|_{aH=k}\,, \tag{11}$$

and so in the single-component case we recover Eq. (2), but with some greater insight into its meaning.

[2]We use the abstract index notation [13]. A boldface superscript index denotes a vector in the scalar field space, a boldface subscript index denotes a covector, contractions are denoted by repeated indices, and the metric $h_{\mathbf{ab}}$ and its inverse $h^{\mathbf{ab}}$ are used to lower and raise indices, respectively.

3 The Calculation

The action during inflation is

$$S = \int d^4x \sqrt{-g} \left[-\frac{1}{2} R + \frac{1}{2} h_{\mathbf{ab}} g^{\mu\nu} (\partial_\mu \phi)^{\mathbf{a}} (\partial_\nu \phi)^{\mathbf{b}} - V(\phi) \right] , \tag{12}$$

where $g_{\mu\nu}$ and R are the metric and curvature scalar in the spacetime, and $h_{\mathbf{ab}}$ is the metric in the scalar field space. The equation of motion for ϕ in the spatially flat homogeneous and isotropic background is

$$\ddot{\phi}^{\mathbf{a}} + 3H\dot{\phi}^{\mathbf{a}} + V^{,\mathbf{a}} = 0 \, , \tag{13}$$

where $\ddot{\phi}^{\mathbf{a}} \equiv D\dot{\phi}^{\mathbf{a}}/dt \equiv \dot{\phi}^{\mathbf{b}} \nabla_{\mathbf{b}} \dot{\phi}^{\mathbf{a}}$ and $\nabla_{\mathbf{a}}$ is the covariant derivative in the scalar field space. We also have

$$3H^2 = V + \frac{1}{2} \dot{\phi}^{\mathbf{a}} \dot{\phi}_{\mathbf{a}} \qquad \text{and} \qquad \dot{H} = -\frac{1}{2} \dot{\phi}^{\mathbf{a}} \dot{\phi}_{\mathbf{a}} \, . \tag{14}$$

We define the slow-roll parameters

$$\epsilon \equiv \frac{1}{2} \frac{|\dot{\phi}|^2}{H^2} \qquad \text{and} \qquad \delta \equiv \frac{\dot{\phi}_{\mathbf{a}} \ddot{\phi}^{\mathbf{a}}}{H |\dot{\phi}|^2} \, , \tag{15}$$

and make the slow-roll assumptions

$$\epsilon = \mathcal{O}(\xi) \qquad \text{and} \qquad \delta = \mathcal{O}(\xi) \, , \tag{16}$$

for some small parameter ξ. We then have

$$\frac{\dot{\epsilon}}{H} = 2\epsilon(\epsilon + \delta) = \mathcal{O}(\xi^2) \tag{17}$$

and we will make the standard extra assumption[3]

$$\frac{\dot{\delta}}{H} = \mathcal{O}(\xi^2) \, . \tag{18}$$

3.1 Equation of motion for the perturbations

The equation of motion for the inflaton perturbation $\delta\phi^{\mathbf{a}}(\mathbf{k}, t)$ on flat hypersurfaces is [8]

$$\frac{D^2 \delta\phi^{\mathbf{a}}}{dt^2} + 3H \frac{D\delta\phi^{\mathbf{a}}}{dt} - R^{\mathbf{a}}{}_{\mathbf{cdb}} \dot{\phi}^{\mathbf{c}} \dot{\phi}^{\mathbf{d}} \, \delta\phi^{\mathbf{b}} + \left(\frac{k}{a} \right)^2 \delta\phi^{\mathbf{a}} + V^{;\mathbf{ab}} \delta\phi_{\mathbf{b}} = \frac{1}{a^3} \frac{D}{dt} \left(\frac{a^3}{H} \dot{\phi}^{\mathbf{a}} \dot{\phi}^{\mathbf{b}} \right) \delta\phi_{\mathbf{b}} \, , \tag{19}$$

where $R^{\mathbf{a}}{}_{\mathbf{bcd}}$ is the curvature tensor in the scalar field space, defined by

$$R^{\mathbf{a}}{}_{\mathbf{bcd}} v^{\mathbf{b}} \equiv \left(\nabla_{\mathbf{c}} \nabla_{\mathbf{d}} - \nabla_{\mathbf{d}} \nabla_{\mathbf{c}} \right) v^{\mathbf{a}} \, . \tag{20}$$

Defining $\varphi^{\mathbf{a}} \equiv a \, \delta\phi^{\mathbf{a}}$, the conformal time $d\eta \equiv dt/a$ and $x \equiv -k\eta$ we get

$$\frac{D^2 \varphi^{\mathbf{a}}}{dx^2} + \left[1 - 2 \left(\frac{aH}{k} \right)^2 \right] \varphi^{\mathbf{a}} = \left(\frac{aH}{k} \right)^2 \left[\frac{\dot{H}}{H^2} h^{\mathbf{ab}} + R^{\mathbf{a}}{}_{\mathbf{cd}}{}^{\mathbf{b}} \frac{\dot{\phi}^{\mathbf{c}}}{H} \frac{\dot{\phi}^{\mathbf{d}}}{H} - \frac{V^{;\mathbf{ab}}}{H^2} + \frac{1}{a^3 H} \frac{D}{H dt} \left(a^3 H \frac{\dot{\phi}^{\mathbf{a}}}{H} \frac{\dot{\phi}^{\mathbf{b}}}{H} \right) \right] \varphi_{\mathbf{b}} \, . \tag{21}$$

[3] For a more general slow-roll approximation which doesn't make this extra assumption, see Ref. [12].

Using

$$x=-k\int\frac{dt}{a}=\frac{k}{aH}\left[1+\epsilon+3\epsilon^2+2\epsilon\delta+\mathcal{O}\left(\xi^3\right)\right] \tag{22}$$

the equation of motion is

$$\frac{D^2\varphi^{\mathbf{a}}}{dx^2}+\left(1-\frac{2}{x^2}\right)\varphi^{\mathbf{a}}=\frac{3}{x^2}\zeta^{\mathbf{a}}{}_{\mathbf{b}}\varphi^{\mathbf{b}} \tag{23}$$

where[4]

$$\begin{aligned}\zeta^{\mathbf{ab}} &= \left(\epsilon+4\epsilon^2+\frac{8}{3}\epsilon\delta\right)h^{\mathbf{ab}}+\left(1+\frac{5}{3}\epsilon\right)\frac{\dot\phi^{\mathbf{a}}}{H}\frac{\dot\phi^{\mathbf{b}}}{H}+\frac{1}{3}\frac{D}{Hdt}\left(\frac{\dot\phi^{\mathbf{a}}}{H}\frac{\dot\phi^{\mathbf{b}}}{H}\right)\\ &\quad+\frac{1}{3}\left(1+2\epsilon\right)R^{\mathbf{a}}{}_{\mathbf{cd}}{}^{\mathbf{b}}\frac{\dot\phi^{\mathbf{c}}}{H}\frac{\dot\phi^{\mathbf{d}}}{H}-\left(1+2\epsilon\right)\frac{V^{;\mathbf{ab}}}{3H^2}+\mathcal{O}\left(\xi^3\right).\end{aligned} \tag{24}$$

3.2 Quantization and Green's function solution

The quantization conditions are

$$\left[\varphi^{\mathbf{a}}(\mathbf{x},\eta),\frac{D\varphi^{\mathbf{b}}}{\partial\eta}(\mathbf{y},\eta)\right]=i\,\delta^{(3)}(\mathbf{x}-\mathbf{y})\,h^{\mathbf{ab}}\,, \tag{25}$$

otherwise zero. With the Fourier transformation

$$\varphi^{\mathbf{a}}(\mathbf{x},\eta)=\int\frac{d^3k}{(2\pi)^{3/2}}\,\varphi^{\mathbf{a}}(\mathbf{k},\eta)\,e^{i\mathbf{k}\cdot\mathbf{x}}\,, \tag{26}$$

the quantization conditions for the Fourier modes are

$$\left[\varphi^{\mathbf{a}}(\mathbf{k},\eta),\frac{D\varphi^{\mathbf{b}\dagger}}{\partial\eta}(\mathbf{l},\eta)\right]=i\,\delta^{(3)}(\mathbf{k}-\mathbf{l})\,h^{\mathbf{ab}}\,, \tag{27}$$

otherwise zero.

Now, the solution of the homogeneous part of Eq. (23),

$$\frac{D^2{\varphi_0}^{\mathbf{a}}}{dx^2}+\left(1-\frac{2}{x^2}\right){\varphi_0}^{\mathbf{a}}=0\,, \tag{28}$$

with the vacuum boundary condition at $x\to\infty$ is

$${\varphi_0}^{\mathbf{a}}(\mathbf{k},x)=\frac{1}{\sqrt{2k}}\left[a^{\mathbf{a}}(\mathbf{k})\,\varphi_0(x)+a^{\dagger\mathbf{a}}(-\mathbf{k})\,\varphi_0^*(x)\right]\,, \tag{29}$$

where

$$\left[a^{\mathbf{a}}(\mathbf{k}),a^{\dagger\mathbf{b}}(\mathbf{l})\right]=\delta^{(3)}(\mathbf{k}-\mathbf{l})\,h^{\mathbf{ab}} \tag{30}$$

and

$$\varphi_0(x)=\left(1+\frac{i}{x}\right)e^{ix}\,. \tag{31}$$

The Green's function solution of Eq. (23) with these boundary conditions is

$$\varphi^{\mathbf{a}}(\mathbf{k},x)={\varphi_0}^{\mathbf{a}}(\mathbf{k},x)+\frac{3i}{2}\int_x^\infty\frac{du}{u^2}\,\zeta^{\mathbf{a}}{}_{\mathbf{b}}(k,u)\,\varphi^{\mathbf{b}}(\mathbf{k},u)\left[\varphi_0^*(u)\,\varphi_0(x)-\varphi_0^*(x)\,\varphi_0(u)\right]. \tag{32}$$

[4] $\zeta^{\mathbf{ab}}$ reduces to $\epsilon^{\mathbf{ab}}$ of Ref. [9] at order ξ, and to $g/3$ of Ref. [6] in the case of a single component inflaton.

3.3 Slow-roll expansion

To implement the standard[5] slow-roll expansion, we expand $\zeta^{\mathbf{ab}}$ in a power series in $\ln x$,

$$\zeta^{\mathbf{ab}} = \sum_{n=0}^{\infty} \zeta_{n+1}^{\mathbf{ab}}(k)\, \frac{(\ln x)^n}{n!}\,, \tag{33}$$

and, in addition to Eqs. (16) and (18), assume

$$\zeta_n{}^{\mathbf{ab}} = \mathcal{O}\left(\xi^n\right)\,. \tag{34}$$

Therefore, to obtain the solution up to second-order corrections, it is sufficient to consider

$$\zeta^{\mathbf{ab}} = \zeta_1{}^{\mathbf{ab}} + \zeta_2{}^{\mathbf{ab}} \ln x\,, \tag{35}$$

where

$$\zeta_1{}^{\mathbf{ab}} = \zeta^{\mathbf{ab}}\Big|_{x=1} = \zeta^{\mathbf{ab}}\Big|_{aH=k} + \mathcal{O}\left(\xi^3\right) \tag{36}$$

$$\begin{aligned} &= \left(1 + 4\epsilon + \frac{8}{3}\delta\right)\epsilon h^{\mathbf{ab}} + \left(1 + \frac{5}{3}\epsilon\right)\frac{\dot\phi^{\mathbf{a}}}{H}\frac{\dot\phi^{\mathbf{b}}}{H} + \frac{1}{3}\frac{D}{Hdt}\left(\frac{\dot\phi^{\mathbf{a}}}{H}\frac{\dot\phi^{\mathbf{b}}}{H}\right) \\ &\quad + \frac{1}{3}\left(1+2\epsilon\right) R^{\mathbf{a}}{}_{\mathbf{cd}}{}^{\mathbf{b}}\frac{\dot\phi^{\mathbf{c}}}{H}\frac{\dot\phi^{\mathbf{d}}}{H} - \left(1+2\epsilon\right)\frac{V^{;\mathbf{ab}}}{3H^2} + \mathcal{O}\left(\xi^3\right)\Big|_{aH=k} \end{aligned} \tag{37}$$

and

$$\zeta_2{}^{\mathbf{ab}} = \frac{D\zeta^{\mathbf{ab}}}{d\ln x}\Big|_{x=1} = -\frac{D\zeta^{\mathbf{ab}}}{Hdt}\Big|_{aH=k} + \mathcal{O}\left(\xi^3\right) \tag{38}$$

$$\begin{aligned} &= -2\epsilon\left(\epsilon+\delta\right)h^{\mathbf{ab}} - \frac{D}{Hdt}\left(\frac{\dot\phi^{\mathbf{a}}}{H}\frac{\dot\phi^{\mathbf{b}}}{H}\right) \\ &\quad - \frac{1}{3}\frac{D}{Hdt}\left(R^{\mathbf{a}}{}_{\mathbf{cd}}{}^{\mathbf{b}}\frac{\dot\phi^{\mathbf{c}}}{H}\frac{\dot\phi^{\mathbf{d}}}{H}\right) + \frac{D}{Hdt}\left(\frac{V^{;\mathbf{ab}}}{3H^2}\right) + \mathcal{O}\left(\xi^3\right)\Big|_{aH=k}\,. \end{aligned} \tag{39}$$

[5] For a more general slow-roll expansion, see Ref. [12].

3.4 Power spectrum

Now, to calculate the power spectrum, we substitute Eq. (35) into Eq. (32) and integrate iteratively. Most of the calculations have already been done in [6], and, in the limit $x \to 0$, the asymptotic form for $\varphi^{\mathbf{a}}$ up to second-order corrections is

$$\varphi^{\mathbf{a}}(\mathbf{k},x) \to \frac{1}{\sqrt{2k}}\frac{i}{x}\left[a^{\mathbf{a}}(\mathbf{k}) - a^{\dagger\mathbf{a}}(-\mathbf{k}) + Z^{\mathbf{a}}{}_{\mathbf{b}}(k,x)\,a^{\mathbf{b}}(\mathbf{k}) - Z^{\mathbf{a}}{}_{\mathbf{b}}{}^{*}(k,x)\,a^{\dagger\mathbf{b}}(-\mathbf{k})\right], \tag{40}$$

where

$$\begin{aligned} Z^{\mathbf{ab}} \;=\;& \left(\alpha + \frac{i\pi}{2}\right)\zeta_1{}^{\mathbf{ab}} + \frac{1}{2}\left[\alpha^2 - \frac{2}{3}\alpha - 4 + \frac{\pi^2}{4} + i\pi\left(\alpha - \frac{1}{3}\right)\right]\zeta_1{}^{\mathbf{a}}{}_{\mathbf{c}}\zeta_1{}^{\mathbf{cb}} \\ & + \frac{1}{2}\left[\alpha^2 + \frac{2}{3}\alpha - \frac{\pi^2}{12} + i\pi\left(\alpha + \frac{1}{3}\right)\right]\zeta_2{}^{\mathbf{ab}} \\ & - \left[\zeta_1{}^{\mathbf{ab}} + \left(\alpha - \frac{1}{3} + \frac{i\pi}{2}\right)\zeta_1{}^{\mathbf{a}}{}_{\mathbf{c}}\zeta_1{}^{\mathbf{cb}} + \frac{1}{3}\zeta_2{}^{\mathbf{ab}}\right]\ln x \\ & + \frac{1}{2}\left(\zeta_1{}^{\mathbf{a}}{}_{\mathbf{c}}\zeta_1{}^{\mathbf{cb}} - \zeta_2{}^{\mathbf{ab}}\right)(\ln x)^2 . \end{aligned} \tag{41}$$

The numerical constant α is defined as

$$\alpha \equiv 2 - \ln 2 - \gamma \simeq 0.729637\,, \tag{42}$$

where $\gamma \simeq 0.577216$ is the Euler-Mascheroni constant. Eq. (40) gives

$$\begin{aligned} &\left\langle \varphi^{\mathbf{a}}(\mathbf{k},x)\,\varphi^{\mathbf{b}\dagger}(\mathbf{l},x)\right\rangle = \frac{1}{2kx^2}\,\delta^{(3)}(\mathbf{k}-\mathbf{l})\left[h^{\mathbf{ab}} + Z^{\mathbf{ab}} + Z^{\mathbf{ba}*} + h_{\mathbf{cd}}Z^{\mathbf{ac}}Z^{\mathbf{bd}*}\right] \\ &= \frac{1}{2kx^2}\,\delta^{(3)}(\mathbf{k}-\mathbf{l})\times \\ &\quad\left\{h^{\mathbf{ab}} + 2\alpha\zeta_1{}^{\mathbf{ab}} + \left(2\alpha^2 - \frac{2}{3}\alpha - 4 + \frac{\pi^2}{2}\right)\zeta_1{}^{\mathbf{a}}{}_{\mathbf{c}}\zeta_1{}^{\mathbf{cb}} + \left(\alpha^2 + \frac{2}{3}\alpha - \frac{\pi^2}{12}\right)\zeta_2{}^{\mathbf{ab}}\right. \\ &\quad\left. - 2\left[\zeta_1{}^{\mathbf{ab}} + \left(2\alpha - \frac{1}{3}\right)\zeta_1{}^{\mathbf{a}}{}_{\mathbf{c}}\zeta_1{}^{\mathbf{cb}} + \frac{1}{3}\zeta_2{}^{\mathbf{ab}}\right]\ln x + \left(2\zeta_1{}^{\mathbf{a}}{}_{\mathbf{c}}\zeta_1{}^{\mathbf{cb}} - \zeta_2{}^{\mathbf{ab}}\right)(\ln x)^2\right\} \end{aligned} \tag{43}$$

and so, from Eq. (7), the power spectrum is

$$\begin{aligned} \mathcal{P}(k) \;=\;& N_{,\mathbf{a}}N_{,\mathbf{b}}\left(\frac{k}{2\pi a x}\right)^2 \times \\ & \left\{h^{\mathbf{ab}} + 2\alpha\zeta_1{}^{\mathbf{ab}} + \left(2\alpha^2 - \frac{2}{3}\alpha - 4 + \frac{\pi^2}{2}\right)\zeta_1{}^{\mathbf{a}}{}_{\mathbf{c}}\zeta_1{}^{\mathbf{cb}} + \left(\alpha^2 + \frac{2}{3}\alpha - \frac{\pi^2}{12}\right)\zeta_2{}^{\mathbf{ab}}\right. \\ & \left. - 2\left[\zeta_1{}^{\mathbf{ab}} + \left(2\alpha - \frac{1}{3}\right)\zeta_1{}^{\mathbf{a}}{}_{\mathbf{c}}\zeta_1{}^{\mathbf{cb}} + \frac{1}{3}\zeta_2{}^{\mathbf{ab}}\right]\ln x + \left(2\zeta_1{}^{\mathbf{a}}{}_{\mathbf{c}}\zeta_1{}^{\mathbf{cb}} - \zeta_2{}^{\mathbf{ab}}\right)(\ln x)^2\right\} \end{aligned} \tag{44}$$

The right hand side is constant to order ξ^2 and so we can choose to evaluate it at any convenient time around horizon crossing. We will evaluate it at $aH = k$.

Now, from Eq. (22),

$$x|_{aH=k} \simeq 1 + \epsilon + 3\epsilon^{\ 2} + 2\epsilon\delta|_{aH=k} \tag{45}$$

Therefore

$$\begin{aligned}\mathcal{P}(k) &= \left(\frac{H}{2\pi}\right)^2 N_{,\mathbf{a}}N_{,\mathbf{b}}\left\{\left(1-2\epsilon-3\epsilon^{\,2}-4\epsilon\delta\right)h^{\mathbf{ab}}+\left[2\alpha-(4\alpha+2)\epsilon\right]\zeta_1{}^{\mathbf{ab}}\right.\\ &\quad \left.\left.+\left(2\alpha^2-\frac{2}{3}\alpha-4+\frac{\pi^2}{2}\right)\zeta_1{}^{\mathbf{a}}{}_{\mathbf{c}}\zeta_1{}^{\mathbf{cb}}+\left(\alpha^2+\frac{2}{3}\alpha-\frac{\pi^2}{12}\right)\zeta_2{}^{\mathbf{ab}}\right\}\right|_{aH=k}, \qquad (46)\end{aligned}$$

where $\zeta_1{}^{\mathbf{ab}}$ and $\zeta_2{}^{\mathbf{ab}}$ are given by Eqs. (36) and (38).

To write $\mathcal{P}(k)$ in terms of the inflaton potential, we define

$$\mathcal{U}\equiv\frac{V^{,\mathbf{a}}}{V}\frac{V_{,\mathbf{a}}}{V}\,,\quad \mathcal{V}_{\mathbf{ab}}\equiv\frac{V_{;\mathbf{ab}}}{V}\,,\quad \mathcal{V}_{\mathbf{abc}}\equiv\mathcal{U}^{\frac{1}{2}}\frac{V_{;\mathbf{abc}}}{V}\,, \qquad (47)$$

the unit covectors

$$e^N_{\mathbf{a}}\equiv\frac{N_{,\mathbf{a}}}{\sqrt{N^{,\mathbf{b}}N_{,\mathbf{b}}}}\quad\text{and}\quad e^V_{\mathbf{a}}\equiv\frac{V_{,\mathbf{a}}}{\sqrt{V^{,\mathbf{b}}V_{,\mathbf{b}}}}\,, \qquad (48)$$

the unit vectors

$$e^{\mathbf{a}}_N\equiv h^{\mathbf{ab}}e^N_{\mathbf{b}}\quad\text{and}\quad e^{\mathbf{a}}_V\equiv h^{\mathbf{ab}}e^V_{\mathbf{b}}\,, \qquad (49)$$

and the component notation

$$w_N\equiv e^{\mathbf{a}}_N w_{\mathbf{a}}\quad\text{and}\quad w_V\equiv e^{\mathbf{a}}_V w_{\mathbf{a}}\,. \qquad (50)$$

Now

$$\frac{V_{,\mathbf{a}}}{V}=\mathcal{U}^{\frac{1}{2}}e^V_{\mathbf{a}} \qquad (51)$$

and using Eqs. (13) and (14) we can derive

$$H^2=\frac{V}{3}\left[1+\frac{1}{6}\mathcal{U}-\frac{1}{12}\mathcal{U}^2+\frac{1}{9}\mathcal{U}\mathcal{V}_{VV}+\mathcal{O}\left(\xi^3\right)\right], \qquad (52)$$

$$\epsilon=\frac{1}{2}\mathcal{U}\left(1-\frac{2}{3}\mathcal{U}+\frac{2}{3}\mathcal{V}_{VV}\right)+\mathcal{O}\left(\xi^3\right), \qquad (53)$$

$$\delta=\frac{1}{2}\mathcal{U}-\mathcal{V}_{VV}+\mathcal{O}\left(\xi^2\right), \qquad (54)$$

$$\frac{\dot\phi_{\mathbf{a}}}{H}\frac{\dot\phi_{\mathbf{b}}}{H}=\left(1-\frac{2}{3}\mathcal{U}\right)\mathcal{U}e^V_{\mathbf{a}}e^V_{\mathbf{b}}+\frac{1}{3}\mathcal{U}\left(e^V_{\mathbf{a}}\mathcal{V}_{\mathbf{b}V}+e^V_{\mathbf{b}}\mathcal{V}_{\mathbf{a}V}\right)+\mathcal{O}\left(\xi^3\right), \qquad (55)$$

$$\frac{D}{Hdt}=-\mathcal{U}^{\frac{1}{2}}\nabla_V-\frac{1}{3}\mathcal{U}^{\frac{1}{2}}h^{\mathbf{ab}}\mathcal{V}_{\mathbf{a}V}\nabla_{\mathbf{b}}+\mathcal{O}\left(\xi^2\right). \qquad (56)$$

The second term in Eq. (56) is needed because, for example, $\mathcal{V}_{\mathbf{a}V\mathbf{b}}=\mathcal{V}_{\mathbf{ab}V}-\mathcal{U}R_{\mathbf{a}VV\mathbf{b}}$ and $\mathcal{U}R_{\mathbf{a}VV\mathbf{b}}$ is of order ξ. Thus, while $\mathcal{V}_{\mathbf{ab}V}$ is of order ξ^2, $\mathcal{V}_{\mathbf{a}V\mathbf{b}}$ is of order ξ. We could drop the second term in Eq. (56) if we assumed either $\mathcal{U}=\mathcal{O}(\xi^2)$, which is physically very reasonable but non-standard, or $R_{\mathbf{a}VV\mathbf{b}}=\mathcal{O}(\xi)$, which is possible but not what one would expect in general.

Expressing Eqs. (46), (36) and (38) in terms of the potential gives

$$\begin{aligned}\mathcal{P}(k) &= \frac{VN_{,\mathbf{e}}N^{,\mathbf{e}}}{12\pi^2}\left\{1-\frac{5}{6}\mathcal{U}-\frac{4}{3}\mathcal{U}^2+\frac{13}{9}\mathcal{U}\mathcal{V}_{VV}+\left[2\alpha-\left(\frac{5}{3}\alpha+1\right)\mathcal{U}\right]\zeta_{1NN}\right.\\ &\left.+\left(2\alpha^2-\frac{2}{3}\alpha-4+\frac{\pi^2}{2}\right)h^{\mathbf{ab}}\zeta_{1N\mathbf{a}}\zeta_{1\mathbf{b}N}+\left(\alpha^2+\frac{2}{3}\alpha-\frac{\pi^2}{12}\right)\zeta_{2NN}\right\}, \qquad (57)\end{aligned}$$

where

$$\begin{aligned}\zeta_{1\mathbf{ab}} &= \frac{1}{2}\left(1+\frac{8}{3}\mathcal{U}-2\mathcal{V}_{VV}\right)\mathcal{U}h_{\mathbf{ab}}+\left(1+\frac{5}{6}\mathcal{U}\right)\left(\mathcal{U}e^V_{\mathbf{a}}e^V_{\mathbf{b}}-\mathcal{V}_{\mathbf{ab}}\right)\\ &+\frac{1}{3}\left(1+\frac{1}{3}\mathcal{U}\right)\mathcal{U}R_{\mathbf{a}VV\mathbf{b}}+\frac{1}{9}\mathcal{U}h^{\mathbf{cd}}\left(R_{\mathbf{a}V\mathbf{cb}}+R_{\mathbf{ac}V\mathbf{b}}\right)\mathcal{V}_{\mathbf{d}V} \qquad (58)\end{aligned}$$

and

$$\begin{aligned}\zeta_{2\mathbf{ab}} &= \mathcal{U}\left(\mathcal{V}_{VV}-\mathcal{U}\right)h_{\mathbf{ab}}-2\mathcal{U}^2e^V_{\mathbf{a}}e^V_{\mathbf{b}}+\mathcal{U}\mathcal{V}_{\mathbf{ab}}+\mathcal{U}\left(e^V_{\mathbf{a}}\mathcal{V}_{\mathbf{b}V}+e^V_{\mathbf{b}}\mathcal{V}_{\mathbf{a}V}\right)-\mathcal{V}_{\mathbf{ab}V}-\frac{1}{3}h^{\mathbf{cd}}\mathcal{V}_{\mathbf{abc}}\mathcal{V}_{\mathbf{d}V}\\ &-\frac{2}{3}\mathcal{U}^2R_{\mathbf{a}VV\mathbf{b}}+\frac{1}{3}\mathcal{U}h^{\mathbf{cd}}\left(R_{\mathbf{a}V\mathbf{cb}}+R_{\mathbf{ac}V\mathbf{b}}\right)\mathcal{V}_{\mathbf{d}V}+\frac{1}{3}\mathcal{U}^{\frac{3}{2}}\left(R_{\mathbf{a}VV\mathbf{b};V}+\frac{1}{3}h^{\mathbf{cd}}R_{\mathbf{a}VV\mathbf{b};\mathbf{c}}\mathcal{V}_{\mathbf{d}V}\right). \qquad (59)\end{aligned}$$

Substituting Eqs. (58) and (59) into Eq. (57), we can obtain the power spectrum in terms of the inflaton potential and its derivatives explicitly. The result is

$$\begin{aligned}\mathcal{P}(k) &= \frac{VN_{,\mathbf{c}}N^{,\mathbf{c}}}{12\pi^2}\left\{1+\left(\alpha-\frac{5}{6}\right)\mathcal{U}+\left(-\frac{1}{2}\alpha^2+\alpha-\frac{17}{6}+\frac{5\pi^2}{24}\right)\mathcal{U}^2\right.\\ &+\left[2\alpha+\left(2\alpha^2-\frac{8}{3}\alpha-9+\frac{7\pi^2}{6}\right)\mathcal{U}\right]\mathcal{U}h^2_{NV}\\ &+\left[-2\alpha+\left(-\alpha^2+\frac{4}{3}\alpha+5-\frac{7\pi^2}{12}\right)\mathcal{U}\right]\mathcal{V}_{NN}\\ &+\left[\frac{2}{3}\alpha+\left(-\alpha-\frac{5}{3}+\frac{2\pi^2}{9}\right)\mathcal{U}\right]\mathcal{U}R_{NVVN}\\ &+\left(\alpha^2-\frac{4}{3}\alpha+\frac{13}{9}-\frac{\pi^2}{12}\right)\mathcal{U}\mathcal{V}_{VV}\\ &+\left(-2\alpha^2+\frac{8}{3}\alpha+8-\frac{7\pi^2}{6}\right)\mathcal{U}h_{NV}\mathcal{V}_{NV}\\ &+\left(-\alpha^2-\frac{2}{3}\alpha+\frac{\pi^2}{12}\right)\left(\mathcal{V}_{NNV}+\frac{1}{3}h^{\mathbf{ab}}\mathcal{V}_{NN\mathbf{a}}\mathcal{V}_{\mathbf{b}V}\right)\\ &+\left(\frac{1}{3}\alpha^2+\frac{2}{9}\alpha-\frac{\pi^2}{36}\right)\mathcal{U}^{\frac{3}{2}}\left(R_{NVVN;V}+\frac{1}{3}h^{\mathbf{ab}}R_{NVVN;\mathbf{a}}\mathcal{V}_{\mathbf{b}V}\right)\\ &+\left(2\alpha^2-\frac{2}{3}\alpha-4+\frac{\pi^2}{2}\right)h^{\mathbf{ab}}\left(\mathcal{V}_{\mathbf{a}N}-\frac{1}{3}\mathcal{U}R_{NVV\mathbf{a}}\right)\left(\mathcal{V}_{\mathbf{b}N}-\frac{1}{3}\mathcal{U}R_{NVV\mathbf{b}}\right)\\ &\left.+\left(\frac{2}{3}\alpha^2+\frac{8}{9}\alpha-\frac{\pi^2}{18}\right)\mathcal{U}h^{\mathbf{ab}}R_{NV\mathbf{a}N}\mathcal{V}_{\mathbf{b}V}+\mathcal{O}\left(\xi^3\right)\right\}. \qquad (60)\end{aligned}$$

Finally, the spectral index, $n-1 \equiv d\ln\mathcal{P}/d\ln k$, and the running of the spectral index can be calculated from Eq. (44) in a similar way. The results are

$$\begin{aligned}
n-1 \;=\; & -\mathcal{U}+\left(2\alpha-\frac{11}{6}\right)\mathcal{U}^2+(-2+\mathcal{U})\,\mathcal{U}h_{NV}^2 \\
& +\left[2+\left(-2\alpha+\frac{1}{3}\right)\mathcal{U}\right]\mathcal{V}_{NN}+\left[-\frac{2}{3}+\left(\frac{4}{3}\alpha+\frac{4}{9}\right)\mathcal{U}\right]\mathcal{U}R_{NVVN} \\
& +\left(-2\alpha+\frac{4}{3}\right)\mathcal{U}\mathcal{V}_{VV}+\left(4\alpha-\frac{8}{3}\right)\mathcal{U}h_{NV}\mathcal{V}_{NV} \\
& +4\alpha\left(\mathcal{U}h_{NV}^2-\mathcal{V}_{NN}+\frac{1}{3}\mathcal{U}R_{NVVN}\right)^2 \\
& +\left(2\alpha+\frac{2}{3}\right)\left(\mathcal{V}_{NNV}+\frac{1}{3}h^{\mathbf{ab}}\mathcal{V}_{NN\mathbf{a}}\mathcal{V}_{\mathbf{b}V}\right) \\
& +\left(-\frac{2}{3}\alpha-\frac{2}{9}\right)\mathcal{U}^{\frac{3}{2}}\left(R_{NVVN;V}+\frac{1}{3}h^{\mathbf{ab}}R_{NVVN;\mathbf{a}}\mathcal{V}_{\mathbf{b}V}\right) \\
& +\left(-4\alpha+\frac{2}{3}\right)h^{\mathbf{ab}}\left(\mathcal{V}_{\mathbf{a}N}-\frac{1}{3}\mathcal{U}R_{NVV\mathbf{a}}\right)\left(\mathcal{V}_{\mathbf{b}N}-\frac{1}{3}\mathcal{U}R_{NVV\mathbf{b}}\right) \\
& +\left(-\frac{4}{3}\alpha-\frac{8}{9}\right)\mathcal{U}h^{\mathbf{ab}}R_{NV\mathbf{a}N}\mathcal{V}_{\mathbf{b}V}+\mathcal{O}\left(\xi^3\right)
\end{aligned} \tag{61}$$

and

$$\begin{aligned}
\frac{dn}{d\ln k} \;=\; & -2\mathcal{U}^2+2\mathcal{U}\mathcal{V}_{NN}-\frac{4}{3}\mathcal{U}^2R_{NVVN}+2\mathcal{U}\mathcal{V}_{VV}-4\mathcal{U}h_{NV}\mathcal{V}_{NV} \\
& -4\left(\mathcal{U}h_{NV}^2-\mathcal{V}_{NN}+\frac{1}{3}\mathcal{U}R_{NVVN}\right)^2 \\
& -2\left(\mathcal{V}_{NNV}+\frac{1}{3}h^{\mathbf{ab}}\mathcal{V}_{NN\mathbf{a}}\mathcal{V}_{\mathbf{b}V}\right) \\
& +\frac{2}{3}\mathcal{U}^{\frac{3}{2}}\left(R_{NVVN;V}+\frac{1}{3}h^{\mathbf{ab}}R_{NVVN;\mathbf{a}}\mathcal{V}_{\mathbf{b}V}\right) \\
& +4h^{\mathbf{ab}}\left(\mathcal{V}_{\mathbf{a}N}-\frac{1}{3}\mathcal{U}R_{NVV\mathbf{a}}\right)\left(\mathcal{V}_{\mathbf{b}N}-\frac{1}{3}\mathcal{U}R_{NVV\mathbf{b}}\right) \\
& +\frac{4}{3}\mathcal{U}h^{\mathbf{ab}}R_{NV\mathbf{a}N}\mathcal{V}_{\mathbf{b}V}+\mathcal{O}\left(\xi^3\right).
\end{aligned} \tag{62}$$

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